

New Online Algorithm for Dynamic Speed Scaling with Sleep State

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Abstract. In this paper, we consider an energy-efficient scheduling problem where n jobs J_1, J_2, \dots, J_n need to be executed such that the total energy usage of these jobs is minimized while ensuring that each job is finished within its deadline. We work in an online setting where a job is known only at its arrival time, along with its processing volume and deadline. In such a setting, the currently best-known algorithm by Han et al. [7] provides a competitive ratio $\max \{4, 2 + \alpha^\alpha\}$ of energy usage. In this paper, we present a new online algorithm SqOA which provides a competitive ratio $\max \{4, 2 + (2 - 1/\alpha)^\alpha 2^{\alpha-1}\}$ of energy usage. For $\alpha \geq 3$, the competitive ratio of our algorithm is better than that of any other existing algorithms for this problem.

Keywords: Energy efficient algorithm, online algorithm, scheduling algorithm

1 Introduction

Over the last few decades, the energy cost for computing has increased exponentially. For example, the electricity cost imposes a significant strain on the budget of data centers where CPUs account for 50 to 60 percent of the energy consumption. The related problems include the heat generated, leading to a reduced reliability of hardware components. As a result, it has become an important challenge for the algorithm researchers to design algorithms which are efficient with respect to the energy usage by the CPU.

Yao, Demers and Shenker [9] were the first one to investigate the following scheduling problem where the objective is to minimize the total energy consumed by the processing of the jobs. The general setting of their problem, on which we also work on, is explained below.

Assume that a processor can run at any arbitrary speed, and consider n jobs J_1, \dots, J_n that have to be processed by such a processor. Each job J_i has a release time r_i , a deadline d_i and a processing volume w_i . A job J_i must be executed in the time interval $[r_i, d_i]$. The processing volume w_i is the amount of work that must be completed to finish J_i . It's also assumed that the preemption of jobs is allowed, i.e. the processing of a job may be stopped and resumed later. If the processor is running at a speed s , the required power $P(s)$ is assumed to be equal to s^α where $\alpha > 1$ is a constant. (In practice, $\alpha = 3$.) The energy

consumed by the execution of the algorithm is the amount of power integrated over the execution period of the algorithm.

Under the above assumptions, Yao et al. [9] developed an $O(n^3)$ -time algorithm (later improved to $O(n^2 \log n)$ -time) to schedule n jobs such that the total energy consumption is minimized. However, this algorithm (called YDS algorithm) works *offline*, i.e., the release time, deadline and the processing volume of each job is known from the beginning. In an online version of the problem, a job J_i is only known at the time of it's arrival, when the deadline d_i and the processing volume w_i are also revealed. An online algorithm is called c -competitive if, for any job sequence, the total energy consumption of the algorithm is at most c times that of an optimal offline algorithm for the same set of jobs.

In the same paper [9] as above, Yao et al. devised two online algorithms for the energy-efficient scheduling problem described above. The first algorithm, called *Average Rate*, has a competitive ratio at most $2^{\alpha-1}\alpha^\alpha$, for any $\alpha \geq 2$. The competitive ratio of the second algorithm, called *Optimal Available (OA)*, is exactly α^α , and therefore the second algorithm is better than the first one in terms of competitiveness. Bansal et al. [5] provided an algorithm BKP which improved the competitive ratio to $2(\frac{\alpha}{\alpha-1})^\alpha e^\alpha$. For small values of α , Bansal et al. [4] provides a better competitive algorithm qOA whose competitiveness is at most $\frac{4^\alpha}{2e^{\frac{1}{2}}\alpha^{\frac{1}{4}}}$.

Let us describe the ideas behind the two algorithms *Optimal Available (OA)* and qOA as these two algorithms are relevant to the work done in this paper. The algorithm OA can be described as follows: whenever a new job arrives, the algorithm computes an optimal schedule for the currently available unfinished jobs. The optimal schedule for the currently available unfinished jobs can be computed using the YDS algorithm mentioned earlier. The algorithm qOA slightly modifies the algorithm OA. It sets the speed of the processor to be $q \geq 1$ times the speed that the algorithm OA would run in the current state. Setting the value of q to be $2 - \frac{1}{\alpha}$, this algorithm achieves a competitive ratio of $\frac{4^\alpha}{2e^{\frac{1}{2}}\alpha^{\frac{1}{4}}}$.

1.1 Our contribution

Irani et al. [8] started the investigation of the scenario where a processor can be transitioned into a sleep state in which the energy consumption is 0. In the active state, the power consumption is given by the equation $P(s) = s^\alpha + g$, where $g > 0$. One can note here that the power consumption is strictly greater than 0 when the processor is in an active state. However, it's not always beneficial to send a processor to the sleep state when it has no work to do, because $L > 0$ amount of wake-up energy is needed to wake-up the processor from the sleep state to the active state. Accordingly, an online algorithm needs to decide when to send a processor to the sleep state, and when to wake-up the processor back to the active state, in addition to determining the speed at every moment during the active state.

Irani et al. [8] devised an online algorithm which is $(2^{2\alpha-2}\alpha^\alpha + 2^{\alpha-1} + 2)$ competitive for energy. Han et al. [7] developed an algorithm SOA which im-

proves the competitive ratio to $\{4, 2 + \alpha^\alpha\}$. In this paper, we combine the ideas used in the algorithms qOA [4] and SOA [7], and provide an algorithm SqOA with a competitive ratio of $\max \{4, 2 + (2 - 1/\alpha)^\alpha 2^{\alpha-1}\}$. Note that our algorithm has a better competitive ratio than SOA for $\alpha \geq 3$. To summarize, we would be proving the following theorem in this paper.

Theorem 1. *SqOA achieves the competitive ratio $\max \{4, 2 + (2 - 1/\alpha)^\alpha 2^{\alpha-1}\}$ for any $\alpha > 1$.*

2 Algorithm SqOA

Let us start with some definitions and notations that we would be using in this paper. A processor is stated to be in a *working state* if it is currently running with a strictly positive speed. The processor is in an *idle state* if the processor is active but running with a speed 0. Finally, the processor is said to be in a *sleep state* if it's no longer active.

Throughout this paper, we denote the notations used in Bansal et al. [4]. The current time is always denoted as t_0 . For $t_0 \leq t' \leq t''$, $w_a(t', t'')$ denotes the total amount of unfinished work for SqOA at t_0 that has a deadline during $(t', t'']$. The quantity $w_o(t', t'')$ is defined similarly for the optimal algorithm OPT. The current speeds of SqOA and OPT are denoted by s_a and s_o , respectively.

Let $d(t', t'') = \max \{0, w_a(t', t'') - w_o(t', t'')\}$ be the excess unfinished work that SqOA has relative to OPT among the already released jobs with deadlines in the range $(t', t'']$, and let ρ denote the current highest density. A sequence of critical times $t_0 < t_1 < t_2 < \dots < t_h$ is defined iteratively as follows: let t_1 be the latest time such that $d(t_0, t_1)/(t_1 - t_0)$ is maximized. Note that t_1 is no more than the latest deadline of any job released thus far. If t_i is earlier than the latest deadline, then let $t_{i+1} > t_i$ be the latest time, not later than the latest deadline, that maximizes $d(t_i, t_{i+1})/(t_{i+1} - t_i)$. We refer to the intervals $[t_i, t_{i+1}]$ as critical intervals. We define g_i as $g_i = d(t_i, t_{i+1})/(t_{i+1} - t_i)$. Note that g_0, g_1, \dots, g_h is a non-negative strictly decreasing sequence.

Let ρ be the current highest density, i.e., $\max_{t > t_0} \frac{w_a(t_0, t)}{(t - t_0)}$. The critical speed s^* is defined as the speed s which minimizes $\frac{P(s)}{s}$. The algorithm SqOA initially starts from a sleep state, and decides on the following actions based on the current state the processor is in.

Case I: The processor is currently in the working state: If $s^* > \rho > 0$, keep working on the job with the earliest deadline at speed s^* . If $\rho \geq s^*$, keep working on the job with the earliest deadline at speed $q\rho$.

Case II: The processor is currently in the idle state: Let $t' \leq t$ be the last time in the working state. (Set $t' = 0$, if undefined). If $\rho \geq s^*$, switch to the working state; otherwise, if $(t - t')g = L$, then switch to the sleep state.

Case III: The processor is currently in the sleep state: If $\rho \geq s^*$, switch

to the working state.

Han et al. [7] proved that if we bound the working energy of their algorithm SOA or any algorithm having same sleep and idle strategy as SOA in terms of working energy of OPT, then that would also bound the total energy (including the idle and wake-up energy) of online algorithm in terms of total energy of OPT. Their proof is based on the property that if SOA wakes up after sleep or idle state then it must be busy till the deadline of the job which the algorithm has started processing after idle/sleep state. Our algorithm SqOA has the same idle and sleep strategy as SOA and so we need to show that the property above also holds for SqOA.

Lemma 1. *Algorithm SqOA is busy till the deadline of the job J which gets processed after idle/sleep interval of SqOA.*

Proof. Note that the processor switches to the working state only when $\rho \geq s^*$. If $\rho = s^*$ at the time t_w of the switching, then the processor is set a speed $q\rho$ at t_w . After an infinitesimal amount of time, the processor is set a speed s^* which is only infinitesimally greater than ρ at that point of time. Therefore, the job J can't finish before it's deadline.

If $\rho > s^*$ at the time t_w of the switching, then the processor is set a speed $q\rho$ at t_w . After a finite amount of time, ρ becomes infinitesimally smaller than s^* and the processor is set a speed s^* . As before, the job J can't finish before it's deadline in this case as well. \square

Based on this observation and the work of Han et al. [7], we can bound the total energy of SqOA by bounding the working energy.

Lemma 2. *If the working energy of SqOA is at most c times that of OPT, then its total energy is at most $\max\{c + 2, 4\}$ times that of OPT.*

We adopt the method of amortized local competitiveness to find the competitive ratio of the algorithm. We choose our potential function as

$$\phi = \beta \sum_{i=0}^{h-1} \hat{g}_i^{\alpha-1} (w_a(t_i, t_{i+1}) - w_o(t_i, t_{i+1}))$$

where $\hat{g}_i = \max\{s^*, g_i\}$. This function $\phi(t)$ is a function of the current time and it must satisfy the following properties.

- $\phi(t)$ is 0 at the arrival of first job and the deadline of last job.
- Its value does not changes abruptly by discrete events like arrival of job, job completion by Optimal or Online algorithm and instantaneous changes in critical intervals.
- At any time t , the following holds: $E_a(t) + \phi(t) \leq cE_o(t)$ where c is the competitiveness ratio.

The following two lemmas are slight modifications of the results proved for the qOA algorithm given by Bansal et al. [4].

Lemma 3. *Job arrival do not change ϕ and also do not change the critical intervals. Also completion of work by OPT or SqOA do not change ϕ .*

Proof. Upon a job arrival, the work of both the online and offline algorithms increase exactly by the same amount, and hence the excess work $d(t', t'')$ does not change for any t' and t'' . Also, ϕ is a continuous function of the unfinished works of SqOA and OPT and the unfinished works continuously decrease to 0. \square

Note that the critical times may change due to SqOA or OPT working on the jobs. So, we must ensure that the instantaneous changes in a critical interval cannot cause an abrupt change in ϕ .

Lemma 4. *Instantaneous changes in critical intervals do not abruptly change ϕ .*

Proof. There are three ways the critical times can change.

- *Merging of two critical intervals:* As SqOA follows an EDF (earliest deadline first) policy, it must work on jobs with deadlines in $[t_0, t_1]$, causing g_0 to decrease until it becomes equal to g_1 . At this point, the critical intervals $[t_0, t_1]$ and $[t_1, t_2]$ merge together. The potential function ϕ does not change by this merger as $g_0 = g_1$ and thus $\hat{g}_0 = \hat{g}_1$ at this point.
- *Splitting of a critical interval:* As OPT works on some job with deadline $t_0 \in (t_k, t_{k+1}]$, the quantity

$$\frac{w_a(t_k, t') - w_o(t_k, t')}{t' - t_k}$$

may increase faster than

$$\frac{w_a(t_k, t_{k+1}) - w_o(t_k, t_{k+1})}{t_{k+1} - t_k},$$

causing this interval to split into two critical intervals, $[t_k, t']$ and $[t', t_{k+1}]$. This split does not change ϕ as the density of the excess work for both of these newly formed intervals is g_k and hence \hat{g}_k is also the same.

- *Formation of a new critical time:* This happens when a job arrives with a deadline later than any of the previous jobs. The creation of the new critical time t_{h+1} doesn't change the potential ϕ because the excess unfinished work $(w_a(t_h, t_{h+1}) - w_o(t_h, t_{h+1}))$ is equal to 0. \square

The observations above imply that the potential function does not change due to any discrete events such as arrivals, job completions, or instantaneous changes in critical intervals. Then, in order to establish that SqOA is c -competitive with respect to energy, it is sufficient to show the following condition at all times:

$$\frac{dE_a(t)}{dt} + \frac{d\phi}{dt} \leq c \frac{dE_o(t)}{dt}.$$

To show this, we need a few more lemmas. Let $s'_o = \max_{t>t_0} \frac{w_o(t_0, t)}{t-t_0}$. Then, the following relationship holds between s'_o , s_o and s^* .

Lemma 5. *If $s_o > 0$ then $s_o \geq s'_o$. Also if $s'_o \geq s^*$ then $s_o \neq 0$.*

Proof. If $s_o < s'_o$, then one of the jobs J contributing to $w_o(t_o, t')$ must be scheduled at a speed greater than s'_o . Also $s_o > 0$ implies that some job (let us call it J') is scheduled at the current time. As the power function $P(s) = s^\alpha + g$ is strictly convex, we can reduce the energy consumption of OPT by increasing the speed of J' , decreasing the speed of J and scheduling it an interval corresponding to J . This contradicts the optimality of OPT.

For the second part of the Lemma, let $s'_o \geq s^*$. If $s_o = 0$, one of the jobs J (among those contributing to $w_o(t_o, t')$) must be scheduled at a speed greater than s'_o and hence with a speed s^* . Note that s^* is the speed at which processing energy of a job is minimized and therefore the following relationship holds: $P(s_1)/s_1 > P(s_2)/s_2 > P(s^*)/s^*$ for $s_1 > s_2 > s^*$. (The inequalities are strict because here P is strictly convex function). By decreasing the speed of J and scheduling it at current time t_o , we can further decrease energy consumption. This contradicts the optimality of OPT. \square

Lemma 6. *If $\rho \geq s^*$, then $s_a \geq qg_0$.*

Proof. By the definition of SqOA, $\rho \geq s^*$ implies that

$$s_a = q \cdot \max_{t>t_0} \frac{w_a(t_0, t)}{t-t_0} \geq q \cdot \frac{w_a(t_0, t_1)}{t_1-t_0} \geq q \cdot \frac{d(t_0, t_1)}{t_1-t_0} = qg_0.$$

Lemma 7. *Suppose $\rho \geq s^*$. If $s_o = 0$, then $s_a \leq qg_0 + qs^*$. If $s_o > 0$, then $s_a \leq qg_0 + qs_o$.*

Proof. By the definition of SqOA, $\rho \geq s^*$ implies that

$$\begin{aligned} s_a &= q \cdot \max_{t>t_0} \frac{w_a(t_0, t)}{t-t_0} \leq q \cdot \max_{t>t_0} \frac{w_o(t_0, t) + d(t_0, t)}{t-t_0} \\ &\leq q \cdot \max_{t>t_0} \frac{w_o(t_0, t)}{t-t_0} + q \cdot \max_{t>t_0} \frac{d(t_0, t)}{t-t_0} \leq q \cdot s'_o + q \cdot g_0. \end{aligned}$$

From Lemma 5, $s_o = 0$ implies that $s'_o < s^*$. Therefore, $s_a \leq qg_0 + qs^*$ in this case. On the other hand, $s_o > 0$ implies that $s'_o \leq s_o$. Therefore, $s_a \leq qg_0 + qs_o$ in this case. \square

We would now move toward proving that the equation $\frac{dE_a(t)}{dt} + \frac{d\phi}{dt} \leq c \frac{dE_o(t)}{dt}$ is true at all times t except at the time of arrival for a job. However, Lemma 3 guarantees that $E_a(t) + \phi(t) \leq cE_o(t)$ is true at all time (including the arrival time of any job). Since ϕ vanishes at the end, the total working energy of SqOA would be at most c times that of OPT. Let us start considering the different cases depending on the values of ρ, g_i, s_o and s_a at the current time. For each of these cases, we would get an inequality in terms of q, β and c . However, we don't know how to choose the values of q, β and c which would satisfy all the equations and minimize c . Instead, we will take $q = 2 - 1/\alpha$, $\beta = c = q^\alpha 2^{\alpha-1}$ and show that all the inequalities are satisfied. Thus, the value of c in the competitive ratio of our algorithm would be $c = q^\alpha 2^{\alpha-1}$.

Without any loss of generality, we can assume that both OPT and SqOA schedule jobs according to the Earliest Deadline First policy, and therefore SqOA is working on a job with a deadline at most t_1 . Let t' be the deadline of the job that OPT is working on, and let k be such that $t_k < t' \leq t_{k+1}$.

Case 1: $g_0 > s^*, k = 0, s_a > 0, s_o > 0$

Since $g_0 > s^*$, there exists an interval $[t_0, t_0 + dt_0]$ such that $\hat{g}_0 = \max\{s^*, g_0\} = g_0$ at any point of time in $[t_0, t_0 + dt_0]$. Therefore,

$$\begin{aligned} \frac{d\phi}{dt_0} &= \beta \frac{d}{dt_0} (g_0^{\alpha-1} (w_a(t_0, t_1) - w_o(t_0, t_1))) \\ &= \beta [g_0^{\alpha-1} (-s_a + s_o) + (\alpha - 1) g_0^{\alpha-2} d(t_0, t_1) \frac{d}{dt_0} \left(\frac{w_a(t_0, t_1) - w_o(t_0, t_1)}{t_1 - t_0} \right)] \\ &= \beta [g_0^{\alpha-1} (-s_a + s_o) + (\alpha - 1) g_0^{\alpha-2} d(t_0, t_1) \left(\frac{(t_1 - t_0)(-s_a + s_o) + d(t_0, t_1)}{(t_1 - t_0)^2} \right)] \\ &= \beta (\alpha g_0^{\alpha-1} (-s_a + s_o) + (\alpha - 1) g_0^\alpha). \end{aligned}$$

As $\frac{dE_a(t)}{dt} = s_a^\alpha + g$ and $\frac{dE_o(t)}{dt} = s_o^\alpha + g$, we need to show that

$$s_a^\alpha + g + \beta (\alpha g_0^{\alpha-1} (-s_a + s_o) + (\alpha - 1) g_0^\alpha) \leq c (s_o^\alpha + g) \quad (1)$$

Equation (1) can be written as

$$s_a^\alpha - \beta \alpha g_0^{\alpha-1} s_a + \beta \alpha g_0^{\alpha-1} s_o + \beta (\alpha - 1) g_0^\alpha - c s_o^\alpha \leq g(c - 1), \text{ which is implied if}$$

$$s_a^\alpha - \beta \alpha g_0^{\alpha-1} s_a + \beta \alpha g_0^{\alpha-1} s_o + \beta (\alpha - 1) g_0^\alpha - c s_o^\alpha \leq 0. \quad (1a)$$

It can be easily seen that $\rho \geq g_0 > s^*$. It follows from Lemma 7 that $qg_0 \leq s_a \leq qg_0 + qs_o$. Note that the left hand side of equation (1a) is a convex function in s_a , and therefore it only remains to prove the inequality for $s_a = qg_0$ and $s_a = q(g_0 + s_o)$.

Substituting $s_a = qg_0$ in the left hand side of (1a), we get

$$q^\alpha g_0^\alpha - \beta q \alpha g_0^\alpha + \beta \alpha g_0^{\alpha-1} s_o + \beta (\alpha - 1) g_0^\alpha - c s_o^\alpha = (q^\alpha - \beta \alpha q + \beta (\alpha - 1)) g_0^\alpha + \beta \alpha g_0^{\alpha-1} s_o - c s_o^\alpha.$$

Taking derivative with respect to s_o , we get that this is maximized at s_o satisfying $c s_o^{\alpha-1} = \beta \alpha g_0^{\alpha-1}$ and hence $s_o = g_0$. Substituting this for s_o and cancelling

g_0^α , it follows that we need to satisfy the following equation: $(q^\alpha - \beta\alpha q + \beta(\alpha - 1)) + \beta(\alpha - 1) \leq 0$. This is easily satisfied as $q^\alpha < \beta$.

Substituting $s_a = qg_0 + qs_o$, the inequality 1(a) becomes

$$\begin{aligned} q^\alpha(g_0 + s_o)^\alpha - \beta\alpha q(g_0 + s_o)g_0^{\alpha-1} + \beta\alpha g_0^{\alpha-1}s_o + \beta(\alpha - 1)g_0^\alpha - cs_o^\alpha &\leq 0 \\ \Leftrightarrow q^\alpha(g_0 + s_o)^\alpha - \beta(q\alpha - (\alpha - 1))g_0^\alpha - \beta\alpha(q - 1)g_0^{\alpha-1}s_o - cs_o^\alpha &\leq 0. \end{aligned} \quad (1b)$$

Let $s_o = xg_0$. It follows from inequality 1(b) that we need to satisfy $q^\alpha(1+x)^\alpha - \beta(q\alpha - (\alpha - 1)) - \beta\alpha(q - 1)x - cx^\alpha \leq 0$. Since $\beta = c = q^\alpha 2^{\alpha-1}$, this inequality would be satisfied if $(1+x)^\alpha - \alpha 2^{\alpha-1} - 2^{\alpha-1}x^\alpha \leq 0$. The maximum value of the left hand side is attained at $x = 1$ where the value of its derivative $\alpha(1+x)^{\alpha-1} - 2^{\alpha-1}\alpha x^{\alpha-1}$ equals 0. It can be easily seen that this maximum value is $(2^\alpha - \alpha 2^{\alpha-1} - 2^{\alpha-1}) = 2^{\alpha-1}(1 - \alpha) < 0$.

Case 2: $g_0 > s^*, k > 0, g_k > s^*, s_a > 0, s_o > 0$

Since $g_0, g_k > s^*$, there exists an interval $[t_0, t_0 + dt_0]$ such that $\hat{g}_0 = \max\{s^*, g_0\} = g_0$ and $\hat{g}_k = \max\{s^*, g_k\} = g_k$ at any point of time in $[t_0, t_0 + dt_0]$. Therefore,

$$\begin{aligned} \frac{d\phi}{dt_0} &= \beta \left[\frac{d}{dt_0} (g_0^{\alpha-1}(w_a(t_0, t_1) - w_o(t_0, t_1))) + \frac{d}{dt_0} (g_k^{\alpha-1}(w_a(t_k, t_{k+1}) - w_o(t_k, t_{k+1}))) \right] \\ &= \beta [\alpha g_0^{\alpha-1}(-s_a) + (\alpha - 1)g_0^\alpha + \alpha g_k^{\alpha-1}(s_o)]. \end{aligned}$$

Note that $g_k < g_0$, as we have observed that g_0, g_1, \dots is a strictly decreasing sequence. Therefore, the quantity $\frac{d\phi}{dt_0}$ above is less than or equal to $\beta(\alpha g_0^{\alpha-1}(-s_a + s_o) + (\alpha - 1)g_0^\alpha)$. Moreover, it can be easily seen that $\frac{dE_a(t)}{dt} = s_a^\alpha + g$ and $\frac{dE_o(t)}{dt} = s_o^\alpha + g$. Therefore, we need to prove exactly the same inequality (1) in case 1.

Case 3: $g_0 > s^*, k > 0, g_k < s^*, s_a > 0, s_o > 0$

Since $g_0, g_k > s^*$, there exists an interval $[t_0, t_0 + dt_0]$ such that $\hat{g}_0 = \max\{s^*, g_0\} = g_0$ and $\hat{g}_k = \max\{s^*, g_k\} = s^*$ at any point of time in $[t_0, t_0 + dt_0]$. Therefore,

$$\begin{aligned} \frac{d\phi}{dt_0} &= \beta \left[\frac{d}{dt_0} (g_0^{\alpha-1}(w_a(t_0, t_1) - w_o(t_0, t_1))) + \frac{d}{dt_0} ((s^*)^{\alpha-1}(w_a(t_k, t_{k+1}) - w_o(t_k, t_{k+1}))) \right] \\ &= \beta [\alpha g_0^{\alpha-1}(-s_a) + (\alpha - 1)g_0^\alpha + (s^*)^{\alpha-1}(s_o)]. \end{aligned}$$

We need to show that $s_a^\alpha + g + \beta[\alpha g_0^{\alpha-1}(-s_a) + (\alpha - 1)g_0^\alpha + (s^*)^{\alpha-1}(s_o)] \leq cs_o^\alpha + cg$. As we have already proven the inequality (1), it suffices to prove that $\beta(s^*)^{\alpha-1}s_o \leq \beta\alpha g_0^{\alpha-1}s_o \Leftrightarrow g_0 \geq \frac{s^*}{\alpha^{\frac{1}{\alpha-1}}}$. This is true since $g_0 > s^*$.

Case 4: $g_0 > s^*, k > 0, g_k = s^*, s_a > 0, s_o > 0$

Since $g_k = s^*$, there exists an interval $[t_0, t_0 + dt_0]$ such that either $\hat{g}_k = \max\{s^*, g_k\} = g_k$, or $\hat{g}_k = \max\{s^*, g_k\} = s^*$ at any point of time in $[t_0, t_0 + dt_0]$.

In the former case, the proof corresponds to case 2 because $\frac{d\phi}{dt}, \frac{dE_a(t)}{dt}, \frac{dE_o(t)}{dt}$ are exactly the same. In the later case, the proof corresponds to Case 3 for the same reason.

Case 5: $g_0 < s^*, s_a > 0, s_o > 0$

As we have observed that g_0, g_1, \dots is a strictly decreasing sequence, $g_0 < s^* \Rightarrow g_k < s^*$. Therefore, $\frac{d\phi}{dt_0}$ remains the same for $k = 0$ and $k > 0$. Hence, we need not consider them as different cases.

As $\frac{d\phi}{dt_0} = \beta(s^*)^{\alpha-1}(-s_a + s_o)$, we need to prove that

$$s_a^\alpha + g + \beta(s^*)^{\alpha-1}(-s_a + s_o) \leq c(s_o^\alpha + g). \quad (2)$$

Consider the case $\rho < s^*$. Then, the algorithm SqOA runs at a speed $s_a = s^*$. So the inequality (2) becomes $(s^*)^\alpha + g + \beta(s^*)^{\alpha-1}(-s^* + s_o) \leq c(s_o^\alpha + g)$.

Let $s_o = xs^*$. Substituting this value of s_o , we obtain

$$\begin{aligned} (s^*)^\alpha + g - \beta(s^*)^\alpha + \beta(s^*)^\alpha x &\leq cx^\alpha(s^*)^\alpha + cg \\ \Leftrightarrow 1 + (\alpha - 1) - \beta + \beta x &\leq cx^\alpha + c(\alpha - 1) \\ \Leftrightarrow \alpha + \beta x &\leq cx^\alpha + c\alpha \\ \Leftrightarrow cx^\alpha + (c - 1)\alpha - cx &\geq 0 \end{aligned}$$

Differentiating the left hand side of the above inequality w.r.t. x and equating it to 0, we obtain $x = (\frac{1}{\alpha})^{\frac{1}{\alpha-1}}$. The value of the left hand side at this value of x is equal to $-c\frac{1}{\alpha^{\frac{1}{\alpha-1}}} + c\alpha - \alpha > -c.1 + c\alpha - \alpha = (c - 1)(\alpha - 1) - 1 > 0$. Since the second derivative of $cx^\alpha + (c - 1)\alpha - cx$ is greater than 0 at any value of x (this is true since $\alpha > 1$), it implies that $cx^\alpha + (c - 1)\alpha - cx$ attains its minimum at $x = (\frac{1}{\alpha})^{\frac{1}{\alpha-1}}$. This completes the proof of inequality (2) in this case.

Next, consider the case $\rho \geq s^*$. From Lemma 7, it implies that $s_a \leq qg_0 + qs_o$. Since $s_a > 0$, the algorithm SqOA is in a working state and it implies that $s_a \geq s^*$. Note that we have already shown that the inequality (2) is true for $s_a = s^*$. As the function $s_a^\alpha + g + \beta(s^*)^{\alpha-1}(-s_a + s_o)$ is convex, it only remains to show that the inequality (2) holds for $s_a = qg_0 + qs_o$ as well.

Consider $s_a = qg_0 + qs_o$. We need to prove that

$$q^\alpha(g_0 + s_o)^\alpha + g + \beta(s^*)^{\alpha-1}(-q(g_0 + s_o) + s_o) \leq cs_o^\alpha + cg.$$

Let $s_o = xg_0$ and $g_0 = ys^*$. (Note that $y \leq 1$.) Substituting this value of s_o and g_0 , we obtain the following inequality which we need to prove.

$$q^\alpha y^\alpha (1 + x)^\alpha - \beta q(1 + x)y + \beta xy \leq cx^\alpha y^\alpha + (c - 1)(\alpha - 1) \quad (3)$$

Let $h(x) = q^\alpha y^\alpha (1+x)^\alpha - cx^\alpha y^\alpha - \beta qy$. Note that the inequality (3) can now be rewritten as $h(x) - \beta qxy + \beta xy \leq (c-1)(\alpha-1)$. Since $(-\beta qxy + \beta xy)$ is negative (it follows from the fact that $q > 1$), it suffices to prove that $h(x) \leq (c-1)(\alpha-1)$.

Differentiating this function $h(x)$, we obtain

$$\begin{aligned} h'(x) &= q^\alpha y^\alpha \alpha (1+x)^{\alpha-1} - cy^\alpha \alpha x^{\alpha-1} \geq 0 \\ \Leftrightarrow (1+x)^{\alpha-1} &\geq 2^{\alpha-1} x^{\alpha-1} \\ \Leftrightarrow x &\leq 1. \end{aligned}$$

Therefore, $h(x)$ attains its maximum at $x = 1$. It can be easily calculated that $h(1) = q^\alpha y^\alpha 2^\alpha - cy^\alpha - \beta qy = q^\alpha y^\alpha (2^\alpha - 2^{\alpha-1}) - \beta qy = q^\alpha y^\alpha 2^{\alpha-1} - \beta qy$. Therefore, the equation (3) will be true if $q^\alpha y^\alpha 2^{\alpha-1} - \beta qy - (c-1)(\alpha-1) \leq 0$.

Note that the equation $q^\alpha y^\alpha 2^{\alpha-1} - \beta qy - (c-1)(\alpha-1)$ is a convex function in y , where $0 \leq y \leq 1$. Therefore, it suffices to show this inequality is true for $y = 0$ and $y = 1$.

If $y = 0$, we need to show that $-(c-1)(\alpha-1) \leq 0$, and if $y = 1$, we need to show that $(q^\alpha 2^{\alpha-1} - \beta q) - (c-1)(\alpha-1) \leq 0$. Both these inequalities are trivially true since $\alpha, c, q > 1$.

Case 6: $s_a > 0, s_o = 0, g_0 > s^*$

Note that $\frac{d\phi}{dt_0} = \beta(\alpha g_0^{\alpha-1}(-s_a) + (\alpha-1)g_0^\alpha)$. Therefore, we need to show that

$$s_a^\alpha + g + \beta(\alpha g_0^{\alpha-1}(-s_a) + (\alpha-1)g_0^\alpha) \leq 0. \quad (4)$$

Since $g_0 > s^*$, it implies that $\rho > s^*$ and hence $qg_0 \leq s_a \leq qg_0 + qs^*$ by Lemma 7.

Consider $s_a = qg_0$. Then, the inequality (4) becomes:

$$\begin{aligned} q^\alpha g_0^\alpha + g - \beta \alpha q g_0^\alpha + \beta(\alpha-1)g_0^\alpha &\leq 0 \\ \Leftrightarrow (q^\alpha - \beta \alpha q + \beta(\alpha-1))g_0^\alpha + g &\leq 0 \\ \Leftrightarrow q^\alpha(1 - \alpha 2^{\alpha-1})g_0^\alpha + g &\leq 0. \end{aligned}$$

The inequality above would be true if $q^\alpha(1 - \alpha 2^{\alpha-1})(s^*)^\alpha + g \leq 0 \Leftrightarrow q^\alpha(1 - \alpha 2^{\alpha-1}) + \alpha - 1 \leq 0$ which can be easily seen to be true.

Next, consider $s_a = qg_0 + qs^*$ and $g_0 = xs^*$. (Note that $x \geq 1$.) Then, the inequality (4) becomes

$$q^\alpha(1+x)^\alpha - \beta \alpha q(1+x)x^{\alpha-1} + \beta(\alpha-1)x^\alpha + \alpha - 1 \leq 0.$$

$$\Leftrightarrow q^\alpha(1+x)^\alpha - \beta\alpha x^\alpha - 2\beta\alpha x^{\alpha-1} + \beta x^{\alpha-1} + \alpha - 1 \leq 0.$$

Since $-2\beta\alpha + \beta < 0 \Rightarrow -2\beta\alpha + \beta \geq -2\beta\alpha x^{\alpha-1} + \beta x^{\alpha-1}$ for $x \geq 1$, we will be done with the proof if $q^\alpha(1+x)^\alpha - \beta\alpha x^\alpha - 2\beta\alpha + \beta + \alpha - 1 \leq 0$.

Let $h(x) = q^\alpha(1+x)^\alpha - \beta\alpha x^\alpha - 2\beta\alpha + \beta + \alpha - 1$. Therefore, we obtain that $h'(x) = \alpha q^\alpha(1+x)^{\alpha-1} - \alpha^2\beta x^{\alpha-1} = 0 \Rightarrow x = \frac{1}{2\alpha^{\frac{1}{\alpha-1}} - 1} < 1$. It shows that $h(x)$ is maximum at $x = 1$ for $x \geq 1$. The proof is completed by observing that $h(1) = q^\alpha 2^\alpha - 3\beta\alpha + \beta + \alpha - 1 = q^\alpha 2^{\alpha-1}(2-3\alpha) + \beta + \alpha - 1 = -q^\alpha 2^{\alpha-1}(3\alpha-1) + \alpha - 1 < 0$.

Case 7: $s_a > 0, s_o = 0, g_0 < s^*$

Observing that $\frac{d\phi}{dt_0} = \beta(s^*)^{\alpha-1}(-s_a)$, we need to show that

$$s_a^\alpha + g - \beta(s^*)^{\alpha-1}s_a \leq 0.$$

Consider the case $\rho < s^*$. In this case, s_a is equal to s^* . On substituting $s_a = s^*$, the above inequality reduces to $\alpha \leq \beta$.

Next, consider the other case $\rho \geq s^*$. From Lemma 7, we obtain $s_a \leq qg_0 + qs^*$. As $s_a \neq 0$, it is greater than or equal to s^* . Therefore, $s^* \leq s_a \leq qg_0 + qs^*$. We have already proven for the case $s^* = s_a$ above. Due to convexity, it only remains to prove the above inequality for $s_a = qg_0 + qs^*$. Substituting $s_a = qg_0 + qs^*$, we can rewrite the inequality above as $q^\alpha(g_0 + s^*)^\alpha + g - \beta(s^*)^{\alpha-1}q(g_0 + s^*) \leq 0$. Let $g_0 = xs^*$. (Note that $x \leq 1$). We can then write the inequality as $q^\alpha(1+x)^\alpha - \beta q(1+x) + \alpha - 1 \leq 0$. As the left hand side of this inequality is a convex function and since $0 \leq x \leq 1$, it suffices to prove the inequality for $x = 0$ and $x = 1$.

For $x = 0$, $q^\alpha - \beta q + \alpha - 1 \leq 0 \Leftrightarrow (q^\alpha - q\beta) + (-q\beta + \alpha - 1) \leq 0$ which can be easily seen. For $x = 1$, $q^\alpha 2^\alpha - 2q\beta + \alpha - 1 \leq 0 \Leftrightarrow (q^\alpha 2^\alpha - 2\beta) - 2(1 - \frac{1}{\alpha})\beta + \alpha - 1 \leq 0 \Leftrightarrow q^\alpha(2^\alpha - 2.2^{\alpha-1}) - 2(1 - \frac{1}{\alpha})\beta + \alpha - 1 \leq 0 \Leftrightarrow -2(1 - \frac{1}{\alpha})\beta + \alpha - 1 \leq 0 \Leftrightarrow \alpha \leq 2\beta$ which can again be seen easily.

Case 8: $s_a = 0, s_o > 0$

Note that the property of algorithm SqOA implies that $s_a = 0 \Rightarrow \rho < s^* \Rightarrow g_0 < s^*$. Since $\frac{dE_a(t)}{dt} = 0$, $\frac{dE_o(t)}{dt} = s_o^\alpha + g$, and $\frac{d\phi}{dt} = \beta(s^*)^{\alpha-1}s_o$, we need to show that the following inequality is true.

$$\beta(s^*)^{\alpha-1}s_o \leq cs_o^\alpha + cg$$

Let $s_o = xs^*$. Then, the above inequality can be rewritten as $x \leq x^\alpha + \alpha - 1 \Leftrightarrow x - x^\alpha \leq \alpha - 1$. Differentiating $x - x^\alpha$, we see that it is maximized at $x = (\frac{1}{\alpha})^{\frac{1}{\alpha-1}}$. It suffices to show that $(\frac{1}{\alpha})^{\frac{1}{\alpha-1}}(1 - \frac{1}{\alpha}) \leq \alpha - 1 \Leftrightarrow (\frac{1}{\alpha})^{\frac{1}{\alpha-1}} \leq \alpha$

which is true since $\alpha > 1$.

Case 9: $s_a = s_o = 0$

As observed above, the property of algorithm SqOA implies that $s_a = 0 \Rightarrow \rho < s^* \Rightarrow g_0 < s^*$. Therefore, $\frac{d\phi}{dt_0} = \beta(s^*)^{\alpha-1}(0 - 0) = 0$. Note that $\frac{dE_a(t)}{dt} = \frac{dE_o(t)}{dt} = 0$ since $s_a = s_o = 0$. It completes the proof for this case.

Case 10: $g_0 = s^*$

Since $g_0 = s^*$, there exists an interval $[t_0, t_0 + dt_0]$ such that either $\hat{g}_0 = \max\{s^*, g_0\} = g_0$ or $\hat{g}_0 = \max\{s^*, g_0\} = s^*$ at any point of time in $[t_0, t_0 + dt_0]$. In the former case, the proof corresponds to one of the cases 1, 2, 3, 4, 6 (depending on the values of s_a, s_o and k) because $\frac{d\phi}{dt}, \frac{dE_a(t)}{dt}, \frac{dE_o(t)}{dt}$ would exactly be the same in the corresponding case. In the later case, the proof corresponds to case 5 or case 7 (again depending on the values of s_a and s_o) for the same reason.

3 Conclusion

In this paper, we have provided a new online algorithm SqOA which improves the competitive ratio for energy from $\{4, 2 + \alpha^\alpha\}$ (Han. et al. [8]) to $\max\{4, 2 + (2 - 1/\alpha)^\alpha 2^{\alpha-1}\}$. We believe that this problem requires further attention, as it is both academically interesting and has practical applications. A particularly interesting research direction would be to analyse SqOA in a multiprocessor environment. Another research direction can be to analyse SqOA in a speed-bounded model.

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